4.9 – Rank, Nullity, and the Fundamental Matrix Spaces

Theorem 4.9.1 The row space and column space of a matrix A have the same dimension. (# of pivots)

Definition: The common dimension of the row space and column space of a matrix A is called the **rank** of A and is denoted by rank(A); the dimension of the null space of A is called the **nullity** of A and is denoted by nullity(A). That is, $\text{rank}(A) = \dim[\text{row}(A)] = \dim[\text{col}(A)]$ and $\text{nullity}(A) = \dim[\text{null}(A)]$.

#1 Find the rank and nullity of the matrix A by reducing it to row echelon form.

a. $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

rank(A) = 1 (one leading 1)

nullity(A) = 3 (3 free variables)

b. $A = \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Rank(A) = 3 (3 leading 1s)

nullity (A) = 2 (2 free variables/parameters)

Theorem 4.9.2 Dimension Theorem for Matrices

If A is a matrix with n columns, then rank(A) + nullity(A) = n.

n: total # variables

rank(A): # leading variables/ # pivots

nullity(A): # free variables/ parameters

Theorem 4.9.3 If A is an $m \times n$ matrix, then

a) rank(A) = the number of leading variables in the general solution of $A\mathbf{x} = \mathbf{0}$.

b) nullity(A) = the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$.

Theorem 4.9.4 If $A\mathbf{x} = \mathbf{b}$ is a consistent linear system of m equations in n unknowns, and if A has rank r, then the general solution of the system contains $n - r$ parameters.

Fundamental spaces of a matrix

Considering a matrix A and its transpose A^T, we have

$$\begin{array}{ll} \text{row}(A) & \text{row}(A^T) \\ \text{col}(A) & \text{col}(A^T) \\ \text{null}(A) & \text{null}(A^T) \end{array}$$

same

same

The four **fundamental spaces** of a matrix A are row (A), col (A), null (A), and null $\left(A^T\right)$. The null space of A^T is also called the **left null space of A**. The **left nullity of A** is nullity $\left(A^T\right) = \dim\left[\text{null }\left(A^T\right)\right]$.

because $A^T\vec{x} = \vec{0}$ becomes $\vec{x}^TA = \vec{0}^T$

Theorem 4.9.5 If A is any matrix, then rank(A) = rank $\left(A^T\right)$.

For an $m \times n$ matrix A,

Since rank(A^T) + nullity$(A^T) = m$, Thm 4.9.5 yields rank(A) + nullity$(A^T) = m$. So

$$\dim[\text{row}(A)] = r \qquad \dim[\text{col}(A)] = r$$

$$\dim[\text{null}(A)] = n - r \qquad \dim[\text{null}(A^T)] = m - r$$

To find bases for the 4 fundamental spaces of an $m \times n$ matrix A, form

$$[A \mid I_m] \xrightarrow{\text{rref}} [R \mid E]$$

- A basis for row(A) is the r rows of R that contain leading 1s (pivots)
- A basis for col(A) is the r columns of A that correspond to pivot columns of R.
- A basis for null (A) is found in the general solution of $A\vec{x} = \vec{0}$
- A basis for null(A^T) – a.k.a. the left null space basis – is the bottom $m-r$ rows of E.

(proof is at the end of this section in the textbook.)

#11 Find the dimensions and bases for the four fundamental spaces of the matrix.

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \\ -9 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|ccc} 1 & 4 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ -9 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|ccc} 1 & 0 & 1 & -4/3 & 0 \\ 0 & 1 & 0 & 1/3 & 0 \\ 0 & 0 & 9 & -12 & 1 \end{array}\right]$$

row(A) basis: $\{(1,0),(0,1)\}$ dim 2

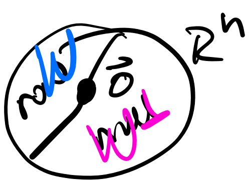
col(A) basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \right\}$ dim 2

null(A^T) basis: $\{[9, -12, 1]\}$ dim 1

$n = 2$ with rank(A) $= 2 \Rightarrow$ nullity(A) $= 0$

null(A) basis: $\emptyset$

Recall (Thm 3.4.3 (which we saw in 3.3)) that if A is an $m \times n$ matrix, then the solution set of the homogeneous system $A\vec{x} = \vec{0}$ consists of all vectors in R^n that are orthogonal to every row vector of A.



row(A) is a subspace of R^n

↓ null(A) (solution space) is a subspace of R^n

$A\vec{x} = \vec{0}$

Definition: If W is a subspace of R^n, then the set of all vectors in R^n that are orthogonal to every vector in W is called the **orthogonal complement** of W and is denoted by the symbol $W^{\perp}$ (pronounced "W perp").

Theorem 4.9.6 If W is a subspace of R^n, then:

a) $W^{\perp}$ is a subspace of R^n.

b) The only vector common to W and $W^{\perp}$ is $\mathbf{0}$, that is, $W \cap W^{\perp} = \{\mathbf{0}\}$.

c) The orthogonal complement of $W^{\perp}$ is W, that is, $(W^{\perp})^{\perp} = W$.

pf: a) $\vec{0} \perp \vec{v}$ $\forall \vec{v} \in \mathbb{R}^n$, so $\vec{0} \perp \vec{w} \in W$

so $W^{\perp}$ is not empty. Let $\vec{w}_1, \vec{w}_2 \in W^{\perp}$.

Then $\vec{w}_1 \perp \vec{w}$ $\forall \vec{w} \in W$ and $\vec{w}_2 \perp \vec{w}$ $\forall \vec{w} \in W$

(def of $W^{\perp}$). Let $\vec{w} \in W$.

$(\vec{w}_1 + \vec{w}_2) \cdot \vec{w} = \vec{w}_1 \cdot \vec{w} + \vec{w}_2 \cdot \vec{w} = 0 + 0 = 0$

$\Rightarrow \vec{w}_1 + \vec{w}_2 \in W^{\perp}$.

Let $k \in \mathbb{R}$. $(k\vec{w}_1) \cdot \vec{w} = k(\vec{w}_1 \cdot \vec{w}) = k(0) = 0$

so $W^{\perp}$ is a subspace of $\mathbb{R}^n$.

Theorem 4.9.7 If A is an $m \times n$ matrix, then: — somewhat updated language for Thm 3.4.3

a) The null space of A and the row space of A are orthogonal complements in R^n.

b) The null space of A^T and the column space of A are orthogonal complements in R^m.

#15 Confirm the orthogonality statements in the two parts of Theorem 4.9.7 for the matrix $A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \\ -9 & 0 \end{bmatrix}$.

Basis for row(A) is $\{(1,0), (0,1)\}$ $\quad$ row(A) $= \mathbb{R}^2$

Basis for col(A) is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \right\} = \{\vec{v}_1, \vec{v}_2\}$

Basis for null(A^T) is $\{(9,-12,1)\} = \{\vec{w}_1\}$

Basis for null(A) is $\emptyset$. $\quad$ null(A) $= \{\vec{0}\}$

Show col(A), null(A^T) are orthogonal complements in $\mathbb{R}^3$.
$\quad$ dim 2 $\quad$ dim 1

$\vec{w}_1 \cdot \vec{v}_1 = 9 - 9 = 0$ ✓

$\vec{w}_1 \cdot \vec{v}_2 = 36 - 36 = 0$ ✓

Side bar: How do we know based on this that $\vec{w} \perp \vec{v}$ for any $\vec{v} \in \text{span}\{\vec{v}_1, \vec{v}_2\}$ and $\vec{w} \in \text{span}\{\vec{w}_1\}$.

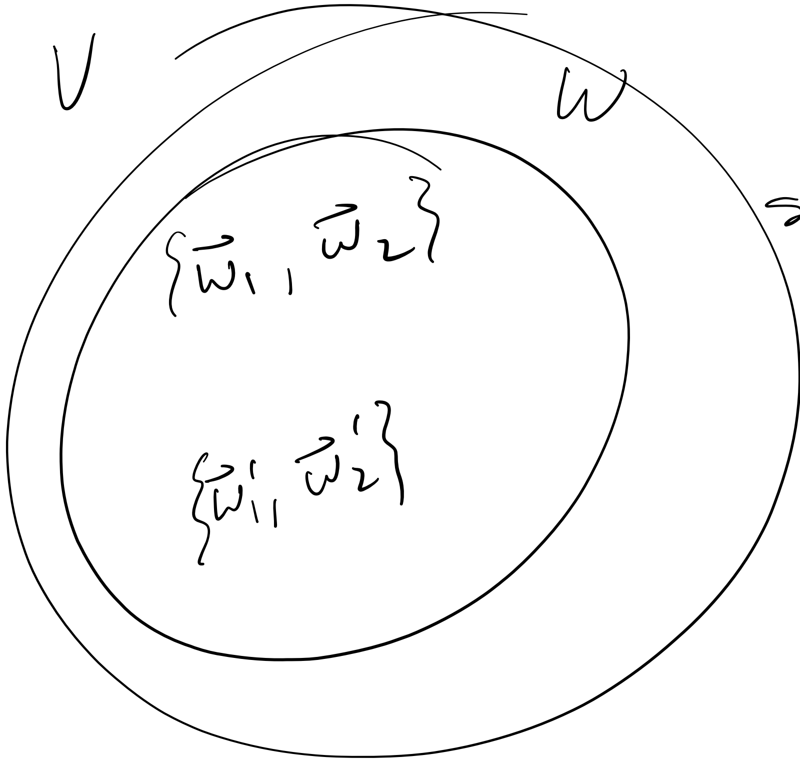
$\vec{v} \in \text{span}\{\vec{v}_1, \vec{v}_2\} \Rightarrow \vec{v} = a\vec{v}_1 + b\vec{v}_2$

$\Rightarrow \vec{w} \cdot \vec{v} = \vec{w} \cdot (a\vec{v}_1 + a\vec{v}_2)$

#27 Suppose that A is a 3×3 matrix whose null space is a line through the origin in 3-space. Can the row or column space of A also be a line through the origin? Explain.

No. The dimensions don't add up to 3.

Theorem 4.9.8 Equivalent Statements (extends Theorem 2.3.8)
If A is an $n \times n$ matrix, then the following statements are equivalent.
a) A is invertible.
b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
c) The reduced row echelon form of A is I_n.
d) A is expressible as a product of elementary matrices.
e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
g) $\det(A) \neq 0$.
h) The column vectors of A are distinct and linearly independent.
i) The row vectors of A are distinct and linearly independent.
j) The column vectors of A span R^n.
k) The row vectors of A span R^n.
l) The column vectors of A form a basis for R^n.
m) The row vectors of A form a basis for R^n.
n) A has rank n.
o) A has nullity 0.
p) The orthogonal complement of the null space of A is R^n.
q) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.

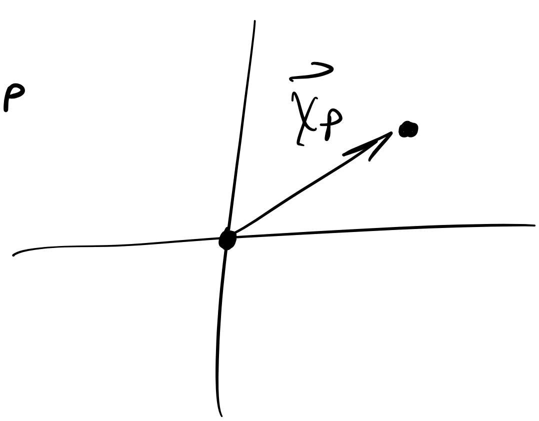


span$\{\vec{w}_1, \vec{w}_2\}$ is a vector space

$$\vec{x} \in \ker(T) \Rightarrow T(\vec{x}) = \vec{0}$$

$$T(\vec{y}+\vec{x}) = T(\vec{y}) + T(\vec{x}) = T(\vec{y})$$

$$\vec{x} = \vec{x}_1 s + \vec{x}_2 t + \vec{x}_p$$



$$\vec{x} = \vec{x}_1 s + \vec{x}_2 t$$

$(s,t) = (0,0)$